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# ON QUADRATIC AND NONQUADRATIC FORMS: APPLICATION TO NONBIJECTIVE $\mathbf{R}^{2m} \to \mathbf{R}^{2m-n}$ TRANSFORMATIONS*

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## Abstract

Hurwitz transformations are defined as specific automorphisms of a Cayley-Dickson algebra. These transformations generate quadratic and nonquadratic forms. We investigate here the Hurwitz transformations corresponding to Cayley-Dickson algebras of dimensions $2m = 2$, 4 and 8. The Hurwitz transformations which lead to quadratic forms are discussed from geometrical and Lie-algebraic points of view. Applications to number theory and dynamical systems are briefly examined.

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# ON QUADRATIC AND NONQUADRATIC FORMS: APPLICATION TO NONBIJECTIVE $\mathbf{R}^{2m} \to \mathbf{R}^{2m-n}$ TRANSFORMATIONS


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## INTRODUCTION

The application of (Hopf) fiber bundles is well developed in theoretical and mathematical physics.[1,2] Along this vein, the Hopf fibrations on spheres lead to nonbijective canonical quadratic transformations useful in classical and quantum mechanics.[3−13] The Hopf fibrations on spheres, as well as their extensions on hyperboloids,[10] yield the concepts of 'constraint Lie algebra' and 'Lie algebra under constraints' which are of importance for connecting invariance or noninvariance algebras of dynamical systems.[11]

On the other hand, there is a close connection between number theory and quadratic mappings, especially those mappings sending the $(2m-1)$-sphere on the $(2m-1-n)$-sphere.[14,15] In this direction, one can easily imagine that the replacement of mappings on spheres by mappings on hyperboloids might produce interesting results.

It is the aim of this work to describe how fibrations on spheres and hyperboloids naturally arise in the framework of Cayley-Dickson algebras. This will provide us with quadratic[10] and nonquadratic[12] forms which generate nonbijective transformations from $\mathbf{R}^{2m}$ onto $\mathbf{R}^{2m-n}$. We shall give two prototypical applications of these quadratic and nonquadratic transformations, viz. one to number theory (in the spirit of Ref. 15) and one to dynamical systems. Much of this work takes its origin in Refs. 5 to 13. The application to number theory constitutes the premises of a more complete work.

## CAYLEY-DICKSON ALGEBRAS

The Cayley-Dickson (C-D) algebras generalize the algebras of complex numbers, quaternions and octonions. A $2m$-dimensional C-D algebra may be obtained from a $m$-dimensional ($m = 1, 2, 4, \cdots$) C-D algebra by a 'doubling' process (cf. $\mathbf{C} = \mathbf{R}+\mathrm{i}\mathbf{R}$). In the present paper, we limit ourselves to C-D algebras of dimensions $2m = 2$, 4 and 8. They are denoted as $A(c)$ with $c \equiv c_1$, $c \equiv c_1, c_2$ and $c \equiv c_1, c_2, c_3$ for $2m = 2$, 4 and 8, respectively, where $c_i = \pm 1$ for $i = 1, 2, 3$. Note that $A(-1)$, $A(-1,-1)$ and $A(-1,-1,-1)$ are nothing but the algebra $\mathbf{C}$ of usual complex numbers, the algebra $\mathbf{H}$ of ordinary (or Hamilton) quaternions and the algebra $\mathbf{O}$ of ordinary (or Cayley) octonions, respectively. The algebra $A(1)$ is the algebra $\mathbf{\Omega}$ of hyperbolic complex numbers. The three algebras $A(c_1, c_2)$ with $(c_1, c_2) \neq (-1,-1)$ are isomorphic to the algebra $\mathbf{N}_1$ of hyperbolic quaternions and the seven algebras $A(c_1, c_2, c_3)$ with $(c_1, c_2, c_3) \neq (-1,-1,-1)$ are isomorphic to the algebra $\mathbf{O}'$ of hyperbolic octonions.

Let

$$u = u_0 + \sum_{k=1}^{2m-1} u_k \, e_k \tag{1}$$

be an element of $A(c)$ where the real numbers $u_0, u_1, \cdots, u_{2m-1}$ are the components of $u$ and the set $\{e_1, e_2, \cdots, e_{2m-1}\}$ is a system of generators of $A(c)$. The product $w = uv$ of the hypercomplex numbers $u$ and $v$ of $A(c)$ is defined once the multiplication rule

$$e_i e_j = -g_{ij} + \sum_{k=1}^{2m-1} a_{ij}^k \, e_k \qquad i \text{ and } j = 1, 2, \cdots, 2m-1 \tag{2}$$

for the generators of $A(c)$ is given. The constants $a_{ij}^k$ (totally antisymmetric in $ijk$) appear in Ref. 10 for $2m = 2$, 4 and 8. The constants $g_{ij}$ are defined by the matrix

$$g \equiv (g_{ij}) = \mathrm{diag}(-c_1, -c_2, c_1 c_2, -c_3, c_1 c_3, c_2 c_3, -c_1 c_2 c_3) \tag{3}$$

for $2m = 8$. The matrix $g$ for $2m = 4$ (or $2m = 2$) follows by restricting the matrix in Eq. (3) to its first four (or two) lines and columns.

The product $w = uv$ can be represented in a matrix form as

$$\mathbf{w} = H(\mathbf{u}; c)\, \mathbf{v} \tag{4}$$

where $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{v}$ are column vectors defined by

$$^t\mathbf{w} = (w_0, w_1, \cdots, w_{2m-1}) \;\; ^t\mathbf{u} = (u_0, u_1, \cdots, u_{2m-1}) \;\; ^t\mathbf{v} = (v_0, v_1, \cdots, v_{2m-1}) \tag{5}$$

and $H(\mathbf{u}; c)$ is a $2m \times 2m$ matrix generalizing the Hurwitz matrix[10] that occurs in the search of bilinerar forms $w_\alpha = \phi_\alpha(u_\beta, v_\beta)$ with $\alpha = 1, 2, \cdots, n$ and $\beta = 1, 2, \cdots, n$ such that

$$w_1^2 + w_2^2 + \cdots + w_n^2 = (u_1^2 + u_2^2 + \cdots + u_n^2)\,(v_1^2 + v_2^2 + \cdots + v_n^2) \tag{6}$$

The matrix $H(\mathbf{u}; c)$ depends on $c$ and the components of $\mathbf{u}$. For $2m = 8$, the matrix

$H(\mathbf{u};c)$ is the following one

$$\begin{pmatrix}
u_0 & c_1u_1 & c_2u_2 & -c_1c_2u_3 & c_3u_4 & -c_1c_3u_5 & -c_2c_3u_6 & c_1c_2c_3u_7 \\
u_1 & u_0 & c_2u_3 & -c_2u_2 & c_3u_5 & -c_3u_4 & c_2c_3u_7 & -c_2c_3u_6 \\
u_2 & -c_1u_3 & u_0 & c_1u_1 & c_3u_6 & -c_1c_3u_7 & -c_3u_4 & c_1c_3u_5 \\
u_3 & -u_2 & u_1 & u_0 & c_3u_7 & -c_3u_6 & c_3u_5 & -c_3u_4 \\
u_4 & -c_1u_5 & -c_2u_6 & c_1c_2u_7 & u_0 & c_1u_1 & c_2u_2 & -c_1c_2u_3 \\
u_5 & -u_4 & -c_2u_7 & c_2u_6 & u_1 & u_0 & -c_2u_3 & c_2u_2 \\
u_6 & c_1u_7 & -u_4 & -c_1u_5 & u_2 & c_1u_3 & u_0 & -c_1u_1 \\
u_7 & u_6 & -u_5 & u_4 & u_3 & u_2 & -u_1 & u_0
\end{pmatrix} \tag{7}$$

The matrix $H(\mathbf{u};c)$ for $2m = 4$ can be obtained from (7) by omitting the 5-, 6-, 7- and 8-th lines and columns. Similarly, the matrix $H(\mathbf{u};c)$ for $2m = 2$ correponds to the two first lines and two first columns of the matrix (7). It can be shown that the matrix $H(\mathbf{u};c)$ for the algebra $A(c)$ can be developed as a linear combination of Clifford matrices. Indeed, we have

$$H(\mathbf{u};c) = u_0\mathbf{1} + \sum_{k=1}^{2m-1} u_k \, {}^t\Gamma_k \tag{8}$$

where $\mathbf{1}$ is the $2m \times 2m$ unit matrix and the $2m \times 2m$ (Clifford) matrices $\Gamma_k$ satisfy

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = -2\, g_{ij}\, \mathbf{1} \qquad (i \text{ and } j = 1, 2, \cdots, 2m-1) \tag{9}$$

It is thus possible to associate a Clifford algebra $\mathcal{C}(p,q)$ of degree $p + q = 2m - 1$ (with $q - p$ being the signature of $g$) to the $2m$-dimensional C-D algebra $A(c)$. An important property of the matrix $H(\mathbf{u};c)$ is the following.

**Property 1**. The matrix $H(\mathbf{u};c)$ belongs to $\mathbf{R}^+ \times \mathrm{O}(2m)$ or $\mathbf{C} \times \mathrm{O}(m,m)$ according to as the $2m \times 2m$ metric

$$\eta = 1 \oplus g \tag{10}$$

has the signature $2m$ of 0. More specifically, we have

$${}^tH(\mathbf{u};c)\, \eta\, H(\mathbf{u};c) = ({}^t\mathbf{u}\, \eta\, \mathbf{u})\, \eta \tag{11}$$

We shall use the qualification compact or noncompact according to as the metric $\eta$ is Euclidean or pseudoEuclidean. The C-D algebra $A(c)$ may be normed in the compact case or pseudonormed (corresponding to a singular division algebra with a cone of zero divisors) in the noncompact case. We close this introductory section with a lemma that is of central importance for what follows.

**Lemma**. The complex conjugation in a $2m$-dimensional C-D algebra $A(c)$ and in its $m$-dimensional C-D subalgebras induces the existence of $2m - \delta(m,1)$ anti-involutions on $A(c)$.

The latter anti-involutions (i.e. involutive anti-automorphisms) $j : A(c) \to A(c)$ can be described in the following way. In the case $m \neq 1$, one anti-involution $j_0$ corresponds to the complex conjugation in the C-D algebra $A(c)$ while the $2m-1$ remaining anti-involutions $j_1, j_2, \cdots, j_{2m-1}$ correspond to the complex conjugation in the various $m$-dimensional C-D subalgebras of $A(c)$. (The complex conjugate $\overline{u}$ of the element $u$ of Eq. (1) is defined in $A(c)$ by $\overline{u} = u_0 - \sum_{k=1}^{2m-1} u_k e_k$.) For $m = 1$, the two-dimensional C-D algebras $A(c_1)$ have only one anti-involution since $j_0$ coincides with $j_1$.

## HURWITZ TRANSFORMATIONS

We associate to the element $u$ of $A(c)$, see Eq. (1), the hypercomplex number

$$\widehat{u} = u_0 + \sum_{k=1}^{2m-1} \varepsilon_k \, u_k \, e_k \tag{12}$$

where $\varepsilon_k = \pm 1$ for $k = 1, 2, \cdots, 2m-1$. The notation

$$\varepsilon = \mathrm{diag}(1, \varepsilon_1, \cdots, \varepsilon_{2m-1}) \tag{13}$$

shall be used below.

**Definition 1**. The mapping

$$T[1; c; \varepsilon] : A(c) \to A(c) : u \mapsto x = u \, \widehat{u} \tag{14}$$

is called a (right) Hurwitz transformation.

The Hurwitz transformations can be classified into several types according to the various possibilities for the $2m \times 2m$ matrix $\varepsilon$.

*Type* $A_1$. For $\widehat{u} = u$: The transformations $T[1; c; \varepsilon]$, where $\varepsilon$ is the unit matrix $\mathbf{1}$, were called quasiHurwitz transformations in Ref. 10 and transformations of type $A_1$ in Ref. 12.

*Type* $B_1$. For $\widehat{u} = j(u)$: The transformations $T[1; c; \varepsilon]$, where $\varepsilon$ corresponds to an arbitrary anti-involution $j$ of $A(c)$, were called Hurwitz transformations in Ref. 10 and transformations of type $B_1$ in Ref. 12.

*Type* $C_1$. For $\widehat{u} \neq u$ or $j(u)$: The transformations $T[1; c; \varepsilon]$, where $\varepsilon$ corresponds neither to the unit matrix $\mathbf{1}$ nor to an anti-involution $j$ of $A(c)$, were called pseudo-Hurwitz transformations in Ref. 10 and transformations of type $C_1$ in Ref. 12.

## QUADRATIC TRANSFORMATIONS

The mapping (14) can be rewritten in matrix form as

$$\mathbf{R}^{2m \times 1} \to \mathbf{R}^{2m \times 1} : \mathbf{u} \mapsto \mathbf{x} = H(\mathbf{u}; c) \, \varepsilon \, \mathbf{u} \tag{15}$$

The column vector $\mathbf{x}$ contains $n$ components that are equal to 0. The integer $n$, which can take several values ($0 \leq n \leq 2m-1$), depends on $\varepsilon$. In other words, the Hurwitz transformation $T[1; c; \varepsilon]$ generates, via Eq. (15), a $\mathbf{R}^{2m} \to \mathbf{R}^{2m-n}$ mapping: To each

vector, of components $u_\alpha$ ($\alpha = 0, 1, \cdots, 2m-1$), in $\mathbf{R}^{2m}$ is associated a vector in $\mathbf{R}^{2m-n}$ the components of which are quadratic functions of the variables $u_\alpha$. Let us emphasize the following property which directly follows from Eq. (11).

**Property 2**. The relation

$$ {}^t\mathbf{x}\,\eta\,\mathbf{x} = \left({}^t\mathbf{u}\,\eta\,\mathbf{u}\right)^2 \tag{16} $$

is valid for all the Hurwitz transformations $T[1; c; \varepsilon]$.

We continue with some examples for $2m = 8$, 4 and 2. The $\mathbf{R}^{2m} \to \mathbf{R}^{2m-n}$ quadratic transformations shall be given in explicit form only for $2m = 8$. The cases $2m = 4$ and $2m = 2$ can be deduced from the case $2m = 8$ by taking $u_4 = u_5 = u_6 = u_7 = 0$ with $c_3 = 0$ and $u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = 0$ with $c_2 = c_3 = 0$, respectively.

*Type $A_1$*. The mapping $T[1; c_1, c_2, c_3; \mathbf{1}]$ gives the generic $\mathbf{R}^8 \to \mathbf{R}^8$ quadratic transformation

$$ x_0 = u_0^2 + c_1 u_1^2 + c_2 u_2^2 - c_1 c_2 u_3^2 + c_3 u_4^2 - c_1 c_3 u_5^2 - c_2 c_3 u_6^2 + c_1 c_2 c_3 u_7^2 $$

$$ x_1 = 2u_0 u_1 \qquad x_2 = 2u_0 u_2 \qquad x_3 = 2u_0 u_3 \qquad x_4 = 2u_0 u_4 $$

$$ x_5 = 2u_0 u_5 \qquad x_6 = 2u_0 u_6 \qquad x_7 = 2u_0 u_7 \tag{17} $$

with the property

$$ x_0^2 - c_1 x_1^2 - c_2 x_2^2 + c_1 c_2 x_3^2 - c_3 x_4^2 + c_1 c_3 x_5^2 + c_2 c_3 x_6^2 - c_1 c_2 c_3 x_7^2 = $$

$$ = (u_0^2 - c_1 u_1^2 - c_2 u_2^2 + c_1 c_2 u_3^2 - c_3 u_4^2 + c_1 c_3 u_5^2 + c_2 c_3 u_6^2 - c_1 c_2 c_3 u_7^2)^2 \tag{18} $$

Equation (17) can be written as

$$ x_0 = 2u_0^2 - {}^t\mathbf{u}\,\eta\,\mathbf{u} \qquad x_k = 2u_0 u_k \qquad (k = 1, 2, \cdots, 7) \tag{19} $$

in condensed form.

From the geometrical point of view, the $\mathbf{R}^{2m} \to \mathbf{R}^{2m}$ transformations ($2m = 8$, 4 and 2) may be arranged in two classes. The compact case corresponds to the fibration $S^{2m-1} \to S^{2m-1}/Z_2 \sim \mathbf{R}P^{2m-1}$ of discrete fiber $Z_2$ while the noncompact case corresponds to the fibration $H^{2m-1}(m,m) \to H^{2m-1}(m,m)/Z_2$ of discrete fiber $Z_2$. (We use $H^{2m-1}(m,m)$ to denote the hyperboloid of equation $\sum_{i=1}^{m} u_i^2 - \sum_{i=m+1}^{2m} u_i^2$ in $\mathbf{R}^{2m}$.)

Note that the transformation $T[1; -1; \mathbf{1}]$ corresponds to the so-called Levi-Civita (conformal) transformation used in the restricted three-body problem.

*Type $B_1$*. The mapping $T[1; c_1, c_2, c_3; \varepsilon]$ where

$$ \varepsilon = \mathrm{diag}(1, -1, 1, 1, 1, 1, -1, -1) \tag{20} $$

gives the generic $\mathbf{R}^8 \to \mathbf{R}^{\mathbf{5}}$ quadratic transformation

$$ x_0 = u_0^2 - c_1 u_1^2 + c_2 u_2^2 - c_1 c_2 u_3^2 + c_3 u_4^2 - c_1 c_3 u_5^2 + c_2 c_3 u_6^2 - c_1 c_2 c_3 u_7^2 $$

$$ x_2 = 2\big(u_0 u_2 + c_1 u_1 u_3 + c_3 u_4 u_6 - c_1 c_3 u_5 u_7\big) $$

$$ x_3 = 2\big(u_0 u_3 + u_1 u_2 - c_3 u_5 u_6 + c_3 u_4 u_7\big) $$

$$x_4 = 2\big(u_0 u_4 + c_1 u_1 u_5 - c_2 u_2 u_6 + c_1 c_2 u_3 u_7\big)$$
$$x_5 = 2\big(u_0 u_5 + u_1 u_4 + c_2 u_3 u_6 - c_2 u_2 u_7\big) \tag{21}$$

with the property

$$x_0^2 - c_2 x_2^2 + c_1 c_2 x_3^2 - c_3 x_4^2 + c_1 c_3 x_5^2 =$$
$$= (u_0^2 - c_1 u_1^2 - c_2 u_2^2 + c_1 c_2 u_3^2 - c_3 u_4^2 + c_1 c_3 u_5^2 + c_2 c_3 u_6^2 - c_1 c_2 c_3 u_7^2)^2 \tag{22}$$

The matrix $\varepsilon$ in Eq. (20) corresponds to an anti-involution of $A(c_1, c_2, c_3)$ of type $j_1, j_2, \cdots, j_7$. The other anti-involutions of type $j_1, j_2, \cdots, j_7$ would lead to transformation formulae equivalent to Eqs. (21) and (22).

The mapping $T[1; c_1, c_2, c_3; \varepsilon]$ where

$$\varepsilon = \mathrm{diag}(1, -1, -1, -1, -1, -1, -1, -1) \tag{23}$$

gives the generic $\mathbf{R}^8 \to \mathbf{R}$ quadratic transformation

$$x_0 = u_0^2 - c_1 u_1^2 - c_2 u_2^2 + c_1 c_2 u_3^2 - c_3 u_4^2 + c_1 c_3 u_5^2 + c_2 c_3 u_6^2 - c_1 c_2 c_3 u_7^2 \tag{24}$$

This transformation (which reads $x_0 = {}^t\mathbf{u}\,\eta\,\mathbf{u}$ in a more compact form) corresponds to the anti-involution $j_0$ of $A(c_1, c_2, c_3)$, i.e. to $\widehat{u} = j_0(u) = \overline{u}$.

**Table 1.** Fibrations (up to homeomorphisms) and Lie algebras under constraints for the $\mathbf{R}^{2m} \to \mathbf{R}^{2m-n}$ quadratic transformations with $n = m - 1 + \delta(m, 1)$.

| Transformation | Fibration | Fiber | $L$ | $L_0$ | $L_1$ |
|---|---|---|---|---|---|
| $\mathbf{R}^2 \to \mathbf{R}$ | $S^1 \to \{1\}$ | $S^1$ | $\mathrm{sp}(4, \mathbf{R})$ | $\mathrm{so}(2)$ | $\mathrm{so}(2,1)$ |
| | $\mathbf{R} \to \{1\}$ | $\mathbf{R}$ | | $\mathrm{so}(1,1)$ | $\mathrm{so}(2,1)$ |
| $\mathbf{R}^4 \to \mathbf{R}^3$ | $S^3 \to S^2$ | $S^1$ | $\mathrm{sp}(8, \mathbf{R})$ | $\mathrm{so}(2)$ | $\mathrm{so}(4,2)$ |
| | $\mathbf{R}^2 \times S^1 \to \mathbf{R}^2$ | $S^1$ | | $\mathrm{so}(2)$ | $\mathrm{so}(4,2)$ |
| | $\mathbf{R}^2 \times S^1 \to \mathbf{R} \times S^1$ | $\mathbf{R}$ | | $\mathrm{so}(1,1)$ | $\mathrm{so}(3,3)$ |
| $\mathbf{R}^8 \to \mathbf{R}^5$ | $S^7 \to S^4$ | $S^3$ | $\mathrm{sp}(16, \mathbf{R})$ | $\mathrm{so}(3)$ | $\mathrm{so}(6,2)$ |
| | $\mathbf{R}^4 \times S^3 \to \mathbf{R}^4$ | $S^3$ | | $\mathrm{so}(3)$ | $\mathrm{so}(6,2)$ |
| | $\mathbf{R}^4 \times S^3 \to \mathbf{R}^2 \times S^2$ | $\mathbf{R}^2 \times S^1$ | | $\mathrm{so}(2,1)$ | $\mathrm{so}(4,4)$ |

The geometrical classification of the $\mathbf{R}^{2m} \to \mathbf{R}^{m+1}$ transformations ($2m = 8$ and 4) associated to an anti-involution of type $j_1, j_2, \cdots, j_{2m-1}$ of $A(c)$ leads to three classes. In the compact case, we have the fibration on spheres $S^{2m-1} \to S^m$ of compact fiber $S^{m-1}$. The noncompact case corresponds to two kinds of fibrations on

hyperboloids, viz. one with a compact fiber and the other one with a noncompact fiber.[10] In addition, the $\mathbf{R}^{2m} \to \mathbf{R}$ transformations ($2m = 8$, 4 and 2) associated to the anti-involution $j_0$ of $A(c)$ may be classified in two classes. The compact case corresponds to the fibration $S^{2m-1} \to \{1\}$ of compact fiber $S^{2m-1}$ and the noncompact case to the fibration $H^{2m-1}(m,m) \to \{1\}$ of noncompact fiber $\mathbf{R}^m \times S^{m-1}$ (up to homeomorphisms). The fibrations corresponding to the anti-involution $j_1$ ($\equiv j_0$) for $2m = 2$ and to the anti-involutions of type $j_1, j_2, \cdots, j_{2m-1}$ for $2m = 4$ and 8 are reported in Table 1.

The case $2m = 4$ deserves three remarks. First, is to be noted that the transformation $T[1; -1, -1; \varepsilon]$ with $\varepsilon = \mathrm{diag}(1, -1, 1, 1)$ can be identified to the so-called Kustaanheimo-Stiefel[3] transformation (associated to the celebrated Hopf fibration on spheres $S^3 \to S^2 \sim S^3/S^1 \sim \mathbf{C}P^1$) used for the regularization of the Kepler problem in classical mechanics. Furthermore, the transformation $T[1; -1, 1; \varepsilon]$ with $\varepsilon = \mathrm{diag}(1, -1, 1, 1)$ corresponds to the transformation introduced by Iwai[7] for reducing an Hamiltonian system by an $S^1$ action. Finally, the transformation $T[1; c_1, c_2; \varepsilon]$ with $\varepsilon = \mathrm{diag}(1, -1, 1, 1)$ and $c_1 = -c_2 = 1$ (or $c_1 = c_2 = 1$) corresponds to a transformation, introduced by Lambert and Kibler,[10] inequivalent to the two previous ones.

*Type* $C_1$. The mapping $T[1; c_1, c_2, c_3; \varepsilon]$, where $\varepsilon$ neither is the unit matrix $\mathbf{1}$ nor corresponds to an anti-involution $j$ of $A(c_1, c_2, c_3)$, yields new quadratic transformations only when $\sum_{k=1}^{7} \varepsilon_k = -3$ or 5. For example, the case

$$\varepsilon = \mathrm{diag}(1, -1, -1, -1, -1, -1, 1, 1) \tag{25}$$

gives the generic $\mathbf{R}^8 \to \mathbf{R}^7$ quadratic transformation

$$\begin{aligned}
x_0 &= u_0^2 - c_1 u_1^2 - c_2 u_2^2 + c_1 c_2 u_3^2 - c_3 u_4^2 + c_1 c_3 u_5^2 - c_2 c_3 u_6^2 + c_1 c_2 c_3 u_7^2 \\
x_2 &= 2(-c_3 u_4 u_6 + c_1 c_3 u_5 u_7) \qquad x_3 = 2(-c_3 u_4 u_7 + c_3 u_5 u_6) \\
x_4 &= 2(c_2 u_2 u_6 - c_1 c_2 u_3 u_7) \qquad x_5 = 2(c_2 u_2 u_7 - c_2 u_3 u_6) \\
x_6 &= 2(u_0 u_6 - c_1 u_1 u_7) \qquad x_7 = 2(u_0 u_7 - u_1 u_6)
\end{aligned} \tag{26}$$

with the property

$$\begin{aligned}
&x_0^2 - c_2 x_2^2 + c_1 c_2 x_3^2 - c_3 x_4^2 + c_1 c_3 x_5^2 + c_2 c_3 x_6^2 - c_1 c_2 c_3 x_7^2 = \\
&= (u_0^2 - c_1 u_1^2 - c_2 u_2^2 + c_1 c_2 u_3^2 - c_3 u_4^2 + c_1 c_3 u_5^2 + c_2 c_3 u_6^2 - c_1 c_2 c_3 u_7^2)^2
\end{aligned} \tag{27}$$

Other choices for $\varepsilon$ with $\sum_{k=1}^{7} \varepsilon_k = -3$ or 5 lead to transformation formulae equivalent to Eqs. (26) and (27). Remark that the restriction of Eq. (26) to the cases $2m = 4$ and $2m = 2$ does not lead to new transformations: The obtained quadratic transformations are Hurwitz transformations of type $B_1$. Furthermore, the (true) transformations of type $C_1$ are equivalent to transformations of type $A_1$ for $2m = 2$ and $2m = 4$.

For $2m = 8$, the $\mathbf{R}^8 \to \mathbf{R}^7$ transformations provide explicit realizations for (i) the Hopf fibration $S^7 \to \mathbf{C}P^3$ of compact fiber $S^1$ when $c_1 + c_2 + c_3 = -3$ and for (ii) its noncompact analogues, namely, $\mathbf{R}^4 \times S^3 \to \mathbf{R}^4 \times S^2$ of compact fiber $S^1$ and $\mathbf{R}^4 \times S^3 \to \mathbf{R}^3 \times S^3$ of noncompact fiber $\mathbf{R}$ when $c_1 + c_2 + c_3 \neq -3$.

## DIFFERENTIAL ASPECTS OF QUADRATIC TRANSFORMATIONS

Differential and Lie-algebraic aspects of transformations of type $A_1$ and $B_1$ were studied in Refs. 10 and 11. Here, we briefly discuss the situation for the $\mathbf{R}^{2m} \to \mathbf{R}^{2m-n}$ quadratic transformations of type $B_1$ with $n = m - 1 + \delta(m, 1)$ and $2m = 2$, 4 and 8.

Let us consider the $2m$ one-forms defined by the column vector

$$\Omega = 2\, H(\mathbf{u}; c)\, \varepsilon\, d\mathbf{u} \tag{28}$$

where the $2m \times 2m$ matrix $\varepsilon$ is associated to a given anti-involution, of $A(c)$, of the type $j_1, j_2, \cdots, j_{2m-1}$. Equation (28) provides us with (i) $2m - n$ total differentials that may be integrated to give the nonvanishing components $x_j$ of $\mathbf{x}$ in Eq. (15) and (ii) $n$ one-forms $\omega_i$ that correspond to the vanishing components of $\mathbf{x}$ in Eq. (15). As a corollary of Property 1, we obtain

$${}^t\Omega\, \eta\, \Omega = 4\, ({}^t\mathbf{u}\, \eta\, \mathbf{u})\, ({}^t d\mathbf{u}\, \eta\, d\mathbf{u}) \tag{29}$$

In view of the nonbijective character of the $\mathbf{R}^{2m} \to \mathbf{R}^{2m-n}$ transformation, we assume that each one-form $\omega_i$ satisfies the constraint condition $\omega_i = 0$. The introduction of these $n$ conditions in Eq. (29) makes it possible to connect the line element in $\mathbf{R}^{2m}$, with the metric $\eta$, and a line element in $\mathbf{R}^{2m-n}$.

To each one-form $\omega_i$, we may associate a vector field $X_i$, defined in the symplectic Lie algebra $\mathrm{sp}(4m, \mathbf{R})$, with an action of $\omega_i$ on $\frac{1}{2r} X_j$, where $r = {}^t\mathbf{u}\eta\mathbf{u}$, such that $\omega_i[\frac{1}{2r} X_j] = \delta(i, j)$. Each vector field $X_i$ has the property that $X_i \psi = 0$ for any function $\psi$ in $C^1(\mathbf{R}^{2m-n})$. The following theorems and definitions are interesting for physical applications.

**Theorem 1**. The $n = m - 1 + \delta(m, 1)$ vector fields $X_i$ span a Lie algebra $L_0$ with respect to the commutator law.

**Definition 2**. The subalgebra $L_0$ of $L = \mathrm{sp}(4m, \mathbf{R})$ is called a *constraint Lie algebra*. Let us define the subalgebra $L_1$ of $L$ by

$$L_1 = \mathrm{cent}_L L_0 / L_0 \tag{30}$$

when $L_0$ is one-dimensional and by

$$L_1 = \mathrm{cent}_L L_0 \tag{31}$$

when $L_0$ is semisimple. The Lie algebra $L_1$ is called a *Lie algebra under constraints*.

**Theorem 2**. The constraint Lie algebra $L_0$ and the Lie algebra under constraints $L_1$ are characterized by the compact or noncompact nature of the fiber of the fibration associated to the corresponding $\mathbf{R}^{2m} \to \mathbf{R}^{m+1-\delta(m,1)}$ quadratic transformation.

The triples $(L, L_0, L_1)$, $L_0 \subset L_1 \subset L$, are reported in Table 1 for the quadratic transformations $\mathbf{R}^2 \to \mathbf{R}$, $\mathbf{R}^4 \to \mathbf{R}^3$ and $\mathbf{R}^8 \to \mathbf{R}^5$.

## NONQUADRATIC TRANSFORMATIONS

Nonquadratic transformations may be obtained by replacing Eq. (15) by

$$\mathbf{R}^{2m\times 1} \to \mathbf{R}^{2m\times 1} : \mathbf{u} \mapsto \mathbf{x} = H(\mathbf{u}; c)^N \, \varepsilon \, \mathbf{u} \quad \text{with} \quad N \in \mathbf{Z} - \{1\} \tag{32}$$

Equations (15) and (32) define $\mathbf{R}^{2m} \to \mathbf{R}^{2m-n}$ quadratic and nonquadratic transformations ($0 \leq n \leq 2m-1$) in a unified way. The property (16) may be generalized as

$$^t\mathbf{x} \, \eta \, \mathbf{x} = (^t\mathbf{u} \, \eta \, \mathbf{u})^{N+1} \tag{33}$$

for the nonquadratic transformations. The quadratic and nonquadratic transformations are associated to Hurwitz transformations that we shall denote as $T[N; c; \varepsilon]$ with $N \in \mathbf{Z}$. They can be classified into transformations of type $A_N$, $B_N$ and $C_N$ by employing the same character of distinction as for the transformations of type $A_1$, $B_1$ and $C_1$. We limit here our consideration to two specific examples: (1) transformations of type $A_N$ with $N \in \mathbf{Z} - \{-1\}$ and $2m = 2$ and (2) transformations of type $B_{-1}$ with $2m = 4$.

**Example 1**. The mapping $T[N; c_1; \mathbf{1}]$ defines the transformations $\mathbf{u} \mapsto \mathbf{x} = H(\mathbf{u}; c_1)^N \, \mathbf{u}$. General properties of these transformations of type $A_N$ ($N \neq -1$) can be easily derived for $2m = 2$. Indeed, we have the five following properties:

$$x_0^2 - c_1 x_1^2 = (u_0^2 - c_1 u_1^2)^{N+1}$$

$$d\mathbf{x} = (N+1) \, H(\mathbf{u}; c_1)^N \, d\mathbf{u}$$

$$dx_0^2 - c_1 dx_1^2 = (N+1)^2 \, (u_0^2 - c_1 u_1^2)^N \, (du_0^2 - c_1 du_1^2)$$

$$\nabla_{\mathbf{x}} = (N+1)^{-1} \, (u_0^2 - c_1 u_1^2)^{-N} \, \eta \, H(\mathbf{u}; c_1)^N \, \eta \, \nabla_{\mathbf{u}}$$

$$\partial_{x_0 x_0} - c_1 \partial_{x_1 x_1} = (N+1)^{-2} \, (u_0^2 - c_1 u_1^2)^{-N} \, (\partial_{u_0 u_0} - c_1 \partial_{u_1 u_1}) \tag{34}$$

which hold outside the domain $\{(u_0, u_1) \in \mathbf{R}^2 \mid u_0^2 - c_1 u_1^2 = 0\}$.

**Example 2**. The mapping $T[-1; c_1, c_2; \varepsilon]$, where $\varepsilon = \mathrm{diag}(1, -1, 1, 1)$, yields the generic nonquadratic formulae

$$x_0 = \frac{1}{\rho^2}\left(u_0^2 + c_1 u_1^2 - c_2 u_2^2 + c_1 c_2 u_3^2\right)$$

$$x_1 = -\frac{2}{\rho^2} u_0 u_1 \qquad x_2 = -\frac{2}{\rho^2} c_1 u_3 u_1 \qquad x_3 = -\frac{2}{\rho^2} u_2 u_1 \tag{35}$$

with $\rho^2 = u_0^2 - c_1 u_1^2 - c_2 u_2^2 + c_1 c_2 u_3^2$. The general property (33) can be particularized as

$$x_0^2 - c_1 x_1^2 - c_2 x_2^2 + c_1 c_2 x_3^2 = 1 \tag{36}$$

Therefore, the nonquadratic transformation of type $B_{-1}$ with $2m = 4$ and $c_1 = c_2 = -1$ corresponds to the $\mathbf{R}^4 \to S^3$ Fock stereographic projection well known in quantum mechanics.

## APPLICATIONS

The applications of the Hurwitz transformations $T[N;c;\varepsilon]$ range from number theory (Hurwitz problem, Pythagorean and Diophantine equations) to theoretical physics (classical and quantum mechanics, gauge theory). We give below a few simple examples of application of some Hurwitz transformations.

### Number Theory

First, the Hurwitz theorem shows that Eq. (6) admits solutions only for $n = 2$, 4 and 8. (The case $n = 1$ is trivial.) The noncompact extension of this theorem concerns the expressions

$$\sum_{\alpha=0}^{m-1} w_\alpha^2 - \sum_{\alpha=m}^{2m-1} w_\alpha^2 = \left(\sum_{\alpha=0}^{m-1} u_\alpha^2 - \sum_{\alpha=m}^{2m-1} u_\alpha^2\right)\left(\sum_{\alpha=0}^{m-1} v_\alpha^2 - \sum_{\alpha=m}^{2m-1} v_\alpha^2\right) \tag{37}$$

for $2m = 2$, 4 and 8. As an interesting result, the solutions $w_\alpha = \phi_\alpha(u_\beta, v_\beta)$ of Eq. (37) are given by $\mathbf{w} = H(\mathbf{u}; c)\mathbf{v}$.

Second, the three generic Diophantine equations

$$A^2 - c_1B^2 - c_2C^2 + c_1c_2D^2 - c_3E^2 + c_1c_3F^2 + c_2c_3G^2 - c_1c_2c_3H^2 = I^2 \tag{38}$$

$$A^2 - c_2B^2 + c_1c_2C^2 - c_3D^2 + c_1c_3E^2 = F^2 \tag{39}$$

and

$$A^2 - c_2B^2 + c_1c_2C^2 - c_3D^2 + c_1c_3E^2 + c_2c_3F^2 - c_1c_2c_3G^2 = H^2 \tag{40}$$

admit solutions $(A, B, \cdots, I) \in \mathbf{Z}^9$, $(A, B, \cdots, F) \in \mathbf{Z}^6$ and $(A, B, \cdots, H) \in \mathbf{Z}^8$ corresponding to quadratic transformations of type $A_1$, $B_1$ and $C_1$, respectively (see Table 2). In Eqs. (38), (39) and (40), each $c_i$ $(i = 1, 2, 3)$ can take the values $+1$ or $-1$ (as in what precedes) in the case $2m = 8$. Then, the solutions are quadratic functions of $u_\alpha \in \mathbf{Z}$ $(\alpha = 0, 1, \cdots, 7)$ given by Eqs. (17) and (18) for (38), Eqs. (21) and (22) for (39) and Eqs. (26) and (27) for (40). Other Diophantine equations are obtained in the cases $2m = 4$ and $2m = 2$ by taking $c_3 = 0$ and $c_2 = c_3 = 0$, respectively.

We have reported in Table 2 some solutions for the Diophantine equations (38), (39) et (40) and their particular cases. In Table 2, the solutions are $(A, B, \cdots, I) \in \mathbf{Z}^9$, $(A, B, \cdots, F) \in \mathbf{Z}^6$ and $(A, B, \cdots, H) \in \mathbf{Z}^8$ for $c_i = \pm 1$ with $i = 1, 2, 3$. The two particular Diophantine equations inside round brackets admit solutions $(A, B, C, D, I) \in \mathbf{Z}^5$ and $(A, B, C, F) \in \mathbf{Z}^4$ corresponding to $u_4 = u_5 = u_6 = u_7$ and $c_3 = 0$ while the Diophantine equation in square brackets admits solutions $(A, B, I) \in \mathbf{Z}^3$ corresponding to $u_2 = u_3 = u_4 = u_5 = u_6 = u_7$ and $c_2 = c_3 = 0$.

For example, for $c_1 = c_2 = c_3 - 1 = -1$ we get from Eq. (39) the Pythagorean equation (see also Table 2)

$$A^2 + B^2 + C^2 = F^2 \tag{41}$$

with the solutions

$$A = u_0^2 + u_1^2 - u_2^2 - u_3^2 \qquad B = 2(u_0u_2 - u_1u_3)$$

$$C = 2(u_0u_3 + u_1u_2) \qquad F = u_0^2 + u_1^2 + u_2^2 + u_3^2 \tag{42}$$

Similarly, for $c_2 = c_3 = 0$ we get from Eq. (38) the Diophantine equation

$$A^2 - c_1 B^2 = I^2 \tag{43}$$

with the solutions

$$A = u_0^2 + c_1 u_1^2 \qquad B = 2u_0 u_1 \qquad I = u_0^2 - c_1 u_1^2 \tag{44}$$

Note that the introduction of specific relations between the $u_\alpha$ in Eqs. (17), (21) and (26) may lead to new Diophantine equations. For instance, by putting $u_1 = 0$ and $u_2 = u_3$ in Eq. (42), we obtain that

$$A = u_0^2 - 2u_2^2 \qquad B = 2u_0 u_2 \qquad F = u_0^2 + 2u_2^2 \tag{45}$$

are solutions of

$$A^2 + 2B^2 = F^2 \tag{46}$$

an equation which comes from Eq. (41).

**Table 2.** Diophantine equations in $\mathbf{Z}^r$ for $r = 3, 4, 5, 6, 8, 9$ with solutions.

| Equation | Solution |
|---|---|
| $A^2 - c_1 B^2 - c_2 C^2$<br>$+ c_1 c_2 D^2 - c_3 E^2 + c_1 c_3 F^2$<br>$+ c_2 c_3 G^2 - c_1 c_2 c_3 H^2 = I^2$<br>$(A^2 - c_1 B^2 - c_2 C^2$<br>$+ c_1 c_2 D^2 = I^2)$<br>$[A^2 - c_1 B^2 = I^2]$ | $A = u_0^2 + c_1 u_1^2 + c_2 u_2^2 - c_1 c_2 u_3^2 + c_3 u_4^2$<br>$- c_1 c_3 u_5^2 - c_2 c_3 u_6^2 + c_1 c_2 c_3 u_7^2$<br>$B = 2u_0 u_1 \quad C = 2u_0 u_2 \quad D = 2u_0 u_3$<br>$E = 2u_0 u_4 \; F = 2u_0 u_5 \; G = 2u_0 u_6 \; H = 2u_0 u_7$<br>$I = u_0^2 - c_1 u_1^2 - c_2 u_2^2 + c_1 c_2 u_3^2$<br>$- c_3 u_4^2 + c_1 c_3 u_5^2 + c_2 c_3 u_6^2 - c_1 c_2 c_3 u_7^2$ |
| $A^2 - c_2 B^2 + c_1 c_2 C^2$<br>$- c_3 D^2 + c_1 c_3 E^2 = F^2$<br>$(A^2 - c_2 B^2 + c_1 c_2 C^2 = F^2)$ | $A = u_0^2 - c_1 u_1^2 + c_2 u_2^2 - c_1 c_2 u_3^2 + c_3 u_4^2$<br>$- c_1 c_3 u_5^2 + c_2 c_3 u_6^2 - c_1 c_2 c_3 u_7^2$<br>$B = 2(u_0 u_2 + c_1 u_1 u_3 + c_3 u_4 u_6 - c_1 c_3 u_5 u_7)$<br>$C = 2(u_0 u_3 + u_1 u_2 - c_3 u_5 u_6 + c_3 u_4 u_7)$<br>$D = 2(u_0 u_4 + c_1 u_1 u_5 - c_2 u_2 u_6 + c_1 c_2 u_3 u_7)$<br>$E = 2(u_0 u_5 + u_1 u_4 + c_2 u_3 u_6 - c_2 u_2 u_7)$<br>$F = u_0^2 - c_1 u_1^2 - c_2 u_2^2 + c_1 c_2 u_3^2$<br>$- c_3 u_4^2 + c_1 c_3 u_5^2 + c_2 c_3 u_6^2 - c_1 c_2 c_3 u_7^2$ |
| $A^2 - c_2 B^2 + c_1 c_2 C^2$<br>$- c_3 D^2 + c_1 c_3 E^2 + c_2 c_3 F^2$<br>$- c_1 c_2 c_3 G^3 = H^2$ | $A = u_0^2 - c_1 u_1^2 - c_2 u_2^2 + c_1 c_2 u_3^2 - c_3 u_4^2$<br>$+ c_1 c_3 u_5^2 - c_2 c_3 u_6^2 + c_1 c_2 c_3 u_7^2$<br>$B = 2(-c_3 u_4 u_6 + c_1 c_3 u_5 u_7)$<br>$C = 2(-c_3 u_4 u_7 + c_3 u_5 u_6)$<br>$D = 2(c_2 u_2 u_6 - c_1 c_3 u_3 u_7)$<br>$E = 2(c_2 u_2 u_7 - c_2 u_3 u_6)$<br>$F = 2(u_0 u_6 - c_1 u_1 u_7)$<br>$G = 2(u_0 u_7 - u_1 u_6)$<br>$H = u_0^2 - c_1 u_1^2 - c_2 u_2^2 + c_1 c_2 u_3^2$<br>$- c_3 u_4^2 + c_1 c_3 u_5^2 + c_2 c_3 u_6^2 - c_1 c_2 c_3 u_7^2$ |

Third, the Hurwitz transformations of type $A_N$, $B_N$ and $C_N$ with $N = 2, 3, 4, \cdots$ lead to Diophantine equations where the variables $A, B, C, \cdots$ are homogeneous polynomials of degree $N+1$ in $u_\alpha$ ($\alpha = 0, 1, \cdots, 2m-1$). As a typical example, from the Hurwitz transformation $T[N; c_1; \mathbf{1}]$ we see that

$$A^2 - c_1 B^2 = C^{N+1} \qquad N = 2, 3, 4, \cdots \tag{47}$$

admit as (particular) solution $A$, $B$ and $C$ such that

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} u_0 & c_1 u_1 \\ u_1 & u_0 \end{pmatrix}^N \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \qquad C = (u_0 \ u_1) \begin{pmatrix} 1 & 0 \\ 0 & -c_1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \tag{48}$$

Particular solutions of more complicated equations generalizing Eq. (47) can be obtained from transformations of type $A_N$, $B_N$ and $C_N$.

## Dynamical Systems

As a second illustration, we end up with a toy example concerning applications of Hurwitz transformations to dynamical systems. Let us consider the two-dimensional Schrvdinger equation

$$\left[ -\frac{1}{2} \left( \partial_{x_0 x_0} + \partial_{x_1 x_1} \right) - \frac{Z}{r^\alpha} \right] \psi = E \psi \qquad \alpha \in \mathbf{R} \tag{49}$$

where $Z \in \mathbf{R}$ and $r = \sqrt{x_0^2 + x_1^2}$. The application to Eq. (49) of a transformation $T[N; -1; \mathbf{1}]$ (of type $A_N$) with $N \in \mathbf{Z} - \{-1\}$ leads to the partial differential equation

$$\left[ -\frac{1}{2} \left( \partial_{u_0 u_0} + \partial_{u_1 u_1} \right) - \left( N+1 \right)^2 E \rho^{2N} \right] \widehat{\psi} = \left( N+1 \right)^2 Z \rho^{2N - \alpha(N+1)} \widehat{\psi} \tag{50}$$

where $\rho = \sqrt{u_0^2 + u_1^2}$ and $\widehat{\psi} \equiv \widehat{\psi}(\mathbf{u})$ is the transform of $\psi \equiv \psi(\mathbf{x})$ under the considered transformation. Furthermore, let us impose that $2N - \alpha(N+1) = 0$. Then, Eq. (50) reduces to

$$\left[ -\frac{1}{2} \left( \partial_{u_0 u_0} + \partial_{u_1 u_1} \right) - \left( N+1 \right)^2 E \rho^{2N} \right] \widehat{\psi} = \left( N+1 \right)^2 Z \widehat{\psi} \tag{51}$$

so that the transformation of type $A_N$ allows to transform the $\mathbf{R}^2$ Schrvdinger equation for the potential $-Z \left( x_0^2 + x_1^2 \right)^{-N/(N+1)}$ and the energy $E$ into the $\mathbf{R}^2$ Schrvdinger equation for the potential $-\left( N+1 \right)^2 E \left( u_0^2 + u_1^2 \right)^N$ and the energy $(N+1)^2 Z$. (Note that in such a transformation the rtles of the energy and the coupling constant are interchanged.) The solutions for $\alpha \in \mathbf{Z}$ and $N \in \mathbf{Z} - \{-1\}$ correspond to the couples $(\alpha, N) = (1, 1)$, $(3, -3)$ and $(4, -2)$. (The solution $(0, 0)$ is trivial !) In other words, the $\mathbf{R}^2$ Schrvdinger equations for the potential $1/r$ (Coulomb potential), $1/r^3$ and $1/r^4$ are transformed into the $\mathbf{R}^2$ Schrvdinger equations for the potentials $\rho^2$ (harmonic oscillator potential), $1/\rho^6$ and $1/\rho^4$, respectively.

To close this paper, let us mention that there exist other applications to dynamical systems. In classical mechanics, the Kustaanheimo-Stiefel transformation (a transformation of type $B_1$) was used for the regularization of the Kepler problem.[3] In quantum mechanics, the latter transformation made it possible to transform the Schrvdinger equation for the three-dimensional hydrogen atom (in an electromagnetic

field) into a Schrvdinger equation for a four-dimensional isotropic harmonic oscillator (with quartic and sextic anharmonic terms) subject to a constraint.[4,5,8]

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